Goats and birds

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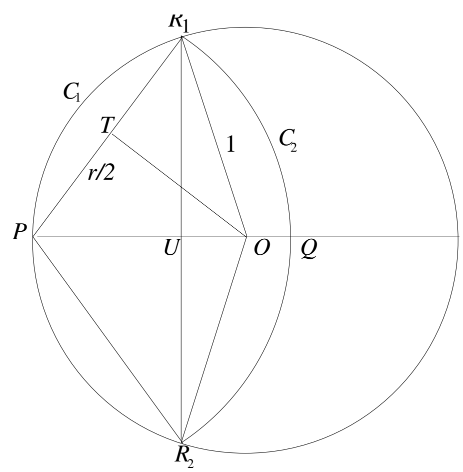
A goatherd wishes to tether a goat to a point on the edge of a circular field, in such a way that it has access to exactly half the area of the field. How long should the rope be? This is usually called the "goat problem", although the creature in question could equally well be a dog, a bull or an alpaca.

Let us take the radius of the field to be 1 unit: it comprises a circle which we denote by C_1, with centre O. The goat is constrained to a circle C_2 of radius r (where r is to be found), with centre at a point P on the circumference of C_1. We need an expression for the area $A(r)$ of the intersection of the two circles. Of course, we take $r \leq 2$ (otherwise C_2 includes the whole of C_1).



In the diagram, let θ_0 be the angle POR_1 and θ_1 the angle OPR_1. Then $OP = OR_1 = 1$ and $PR_1 = r$. Clearly, $2\theta_1 = \pi - \theta_0$, and from the triangle OPT we see that

$$\frac{r}{2} = \cos\theta_1 = \sin\tfrac{1}{2}\theta_0, \tag{1}$$

so that

$$\theta_1 = \cos^{-1}\frac{r}{2}, \qquad \theta_0 = 2\sin^{-1}\frac{r}{2}. \tag{2}$$

The area of the sector OR_1PR_2 of C_1 is θ_0, and the area of the sector PR_1QR_2 of C_2 is $r^2\theta_1$. If we add these two areas, we have counted the triangles OPR_1 and OPR_2 twice. Now the area of OPR_1 is clearly $\cos\theta_1 \sin\theta_1$, so we have, as a first version,

$$A(r) = \theta_0 + r^2\theta_1 - 2\cos\theta_1\sin\theta_1. \tag{3}$$

Our diagram shows the case when $\theta_0 \leq \frac{\pi}{2}$, so that $r \leq \sqrt{2}$, but the expression remains valid when $\theta_0 > \frac{\pi}{2}$. It has arisen in a completely natural way, each term representing a recognisable area, but as it stands, it mixes r, θ_0 and θ_1, each of which can be expressed in terms of the others. What we really want, of course, is $A(r)$ in terms of r: this is easily achieved by substitution from (1) and (2):

$$A(r) = 2\sin^{-1}\frac{r}{2} + r^2\cos^{-1}\frac{r}{2} - r\left(1 - \frac{r^2}{4}\right)^{1/2}. \tag{4}$$

An equivalent version of this formula can be seen on the website [1].

It is worth recording two particularly simple special cases:

$$A(1) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}, \qquad A(\sqrt{2}) = \pi - 1,$$

so in the case $r = \sqrt{2}$, the area of the remaining crescent-shaped part of C_1 is exactly 1.

However, to settle the goatherd's probem, we have to solve the equation $A(r) = \frac{\pi}{2}$. Of course, there is no question of an analytic solution. Solving numerically by the Newton-Raphson method, we find $r = 1.158728473$ to ten significant figures.

Before leaving this problem, we revisit the formula for the area $A(r) = A$, now expressing it in terms of θ_1. Substituting in (3), we see that

$$\begin{aligned} A &= \pi - 2\theta_1 + 4\theta_1\cos^2\theta_1 - 2\cos\theta_1\sin\theta_1 \\ &= \pi + 2\theta_1\cos 2\theta_1 - \sin 2\theta_1. \end{aligned} \tag{5}$$

As well as being simpler than (4), this expression is more pleasant to use for the Newton-Raphson iteration.

We now describe an interesting alternative derivation of (5) by calculus. Geometrically, it is clear that $\frac{dA}{dr}$ equates to the length of the arc R_1QR_2, that is, $2r\theta_1$. (One can verify that this is implied by (4), after a good deal of cancellation). Since $r = 2\cos\theta_1$, we have

$$\frac{dA}{d\theta_1} = \frac{dA}{dr}\frac{dr}{d\theta_1} = -8\theta_1\cos\theta_1\sin\theta_1 = -4\theta_1\sin 2\theta_1.$$

When $\theta_1 = 0$, we have $r = 2$ and hence $A = \pi$. Now writing θ for θ_1 and integrating from 0 to θ_1, we obtain

$$\begin{aligned} A - \pi &= -4\int_0^{\theta_1} \theta\sin 2\theta\, d\theta \\ &= \Big[2\theta\cos 2\theta\Big]_0^{\theta_1} - \int_0^{\theta_1} 2\cos 2\theta\, d\theta \\ &= 2\theta_1\cos 2\theta_1 - \sin 2\theta_1, \end{aligned}$$

agreeing with (5).

Now let us turn to the analagous three-dimensional problem. In a spherical cage of radius 1, a bird is to be tethered to a point on the side of the cage in such a way that it has access to exactly half the volume of the cage. It seems natural to call this the "bird problem", but this does not seem to be established terminology.

We now have two spheres S_1 (radius 1) and S_2 (radius r), and we have to evaluate the volume $V(r)$ of their intersection. The most elementary way to do this is to treat it as a volume of revolution. It is comprised of slices V_1, V_2 of the two spheres cut off by the plane where the surfaces intersect. First, consider the volume of a slice of a sphere in general. Let the sphere have radius r, and slice it by a plane at distance b from the centre. This slice is the volume of revolution obtained by rotating the circular arc $y = (r^2 - x^2)^{1/2}$ $(b \le x \le r)$ around the x-axis, so its volume is

$$\begin{aligned} V &= \pi \int_b^r (r^2 - x^2)\, dx \\ &= \pi r^2(r-b) - \frac{\pi}{3}(r^3 - b^3) \\ &= \frac{\pi}{3}(r-b)(2r^2 - rb - b^2) \\ &= \frac{\pi}{3}(r-b)^2(2r+b). \end{aligned}$$

In our case, the slice V_1 (of S_1) is at distance $OU = \cos\theta_0$ from O (the two-dimensional diagram still serves). By (1), $\cos\theta_0 = 1 - 2\sin^2 \frac{1}{2}\theta_0 = 1 - \frac{r^2}{2}$, so

$$V_1 = \frac{\pi}{3}.\frac{r^4}{4}\left(3 - \frac{r^2}{2}\right) = \frac{\pi}{24}(6r^4 - r^6). \tag{6}$$

The slice V_2 (of S_2) is at distance $PU = r\cos\theta_1 = \frac{r^2}{2}$ from the centre P, so

$$\begin{aligned} V_2 &= \frac{\pi}{3}\left(r - \frac{r^2}{2}\right)^2\left(2r + \frac{r^2}{2}\right) \\ &= \frac{\pi}{24}(2r - r^2)^2(4r + r^2) \\ &= \frac{\pi}{24} r^3(2-r)^2(4+r) \\ &= \frac{\pi}{24} r^3(16 - 12r + r^3). \end{aligned} \tag{7}$$

By (6) and (7),

$$V(r) = V_1 + V_2 = \frac{\pi}{24}(16r^3 - 6r^4) = \frac{\pi}{12}(8r^3 - 3r^4), \tag{8}$$

a formula that might seem a bit surprising, but a distinctly simpler one than the expression (4) for $A(r)$.

For the bird problem, we equate $V(r)$ to half the volume of S_1, i.e. $\frac{2}{3}\pi$, obtaining the equation $8r^3 - 3r^4 = 8$. With enough determination, one can derive an explicit algebraic solution to this quartic equation:

$$r = \frac{2}{3} + \frac{1}{2}\left(\sqrt{a} - \sqrt{\frac{16}{3} - a + \frac{128}{27\sqrt{a}}}\right), \tag{9}$$

where

$$a = \frac{16}{9} + \frac{4}{3}\left((4+\sqrt{8})^{1/3} + (4-\sqrt{8})^{1/3}\right).$$

However, some bird handlers might prefer a numerical answer. Such an answer is delivered more readily by the Newton-Raphson method (once again) rather than by (9): we find that $r = 1.228544864$ to ten s.f.

Some mathematicians, of course, find it altogether too prosaic to be bound by three dimensions. The n-dimensional version of the goat problem has indeed been addressed. The results are given in [2] and [3] (we are grateful to Nick Lord for these references). Our expression (8) duly appears as a special case, and it transpires that as n tends to infinity, the required length of rope converges to $\sqrt{2}$ (actually, this is asserted in [2], but the proof has a gap which is rectified in [3]).

References

1. Eric W. Weisstein, Goat problem, at mathworld.wolfram.com/GoatProblem.html
2. Marshall Fraser, The Grazing Goat in n dimensions, *College Math. J.* **15** (1984), 126–134.
3. Mark D. Meyerson, Return of the Grazing Goat in n dimensions, *College Math. J.* **15** (1984), 430–432.

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