

Three answers to an integral

Problem: By a suitable substitution, find

$$\int \frac{1}{\cosh x} dx.$$

Student 1 (and in this case, the lecturer) did the following. We have

$$\frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}.$$

Substitute $u = e^x$, or $x = \ln u$, so that $dx/du = 1/u$. The integral becomes

$$\int \frac{2}{u + 1/u} \frac{1}{u} du = \int \frac{2}{u^2 + 1} du = 2 \tan^{-1} u + c = 2 \tan^{-1} e^x + c.$$

Student 2 started by writing

$$\frac{1}{\cosh x} = \frac{\cosh x}{\cosh^2 x} = \frac{\cosh x}{1 + \sinh^2 x}.$$

The “suitable” substitution is now $u = \sinh x$, so that $\cosh x dx = du$ and the integral becomes

$$\int \frac{1}{1 + u^2} du = \tan^{-1} u + c = \tan^{-1} \sinh x + c.$$

Who is right? Both are! But are the answers the same? Of course, this doesn't mean that $\tan^{-1} \sinh x$ is *equal* to $2 \tan^{-1} e^x$, but only that they differ by a constant. When $x = 0$, the first function equals 0 and the second one equals $2 \tan^{-1} 1 = \pi/2$, so the only candidate for the constant is $\pi/2$. Equality, with this constant inserted, necessarily follows from the fact that the two functions have the same derivative, but it is of some interest to prove it directly by elementary means, as follows.

Proposition. For $u > 0$,

$$\tan^{-1} \left[\frac{1}{2} \left(u - \frac{1}{u} \right) \right] = 2 \tan^{-1} u - \frac{\pi}{2}, \quad (1)$$

and hence for any x ,

$$\tan^{-1} \sinh x = 2 \tan^{-1} e^x - \frac{\pi}{2}. \quad (2)$$

Proof. Recall that $\tan^{-1} y$ is defined to be the (unique) x in the interval $(-\pi/2, \pi/2)$ such that $\tan x = y$. As already remarked above, (1) holds when $u = 1$. Suppose that $u \neq 1$, and let $a = \tan^{-1} u$, so that $\tan a = u$ and $0 < a < \pi/2$. Then (1) can be written as

$$2a - \pi/2 = \tan^{-1} \left[\frac{1}{2} \left(u - \frac{1}{u} \right) \right]. \quad (3)$$

Clearly, $2a - \pi/2$ is in the right interval $(-\pi/2, \pi/2)$. Also, the formula for $\tan 2a$ gives

$$\tan 2a = \frac{2u}{1-u^2}$$

(note that $u \neq 1$), and hence

$$\tan\left(2a - \frac{\pi}{2}\right) = -\cot 2a = \frac{u^2 - 1}{2u} = \frac{1}{2}\left(u - \frac{1}{u}\right).$$

So (3) is true.

What about the integral of $1/(\sinh x)$? The reader is invited to check that the substitutions $u = e^x$ and $u = \cosh x$ lead, respectively, to

$$\ln|e^x - 1| - \ln(e^x + 1) + c$$

and

$$\frac{1}{2}\ln(\cosh x - 1) - \frac{1}{2}\ln(\cosh x + 1) + c,$$

and to reconcile these solutions by checking that

$$\left(\frac{e^x - 1}{e^x + 1}\right)^2 = \frac{\cosh x - 1}{\cosh x + 1}.$$

For this integral, an interesting *third* answer is delivered by the substitution $u = 1/\sinh x$. Then $x = \sinh^{-1} 1/u$, hence (for $u > 0$),

$$\frac{dx}{du} = \frac{1}{(1 + 1/u^2)^{1/2}} \left(-\frac{1}{u^2}\right) = -\frac{1}{u(u^2 + 1)^{1/2}},$$

so that (for $x > 0$)

$$\int \frac{1}{\sinh x} dx = -\int \frac{1}{(u^2 + 1)^{1/2}} du = -\sinh^{-1} u = -\sinh^{-1} \frac{1}{\sinh x} + c.$$

For $x < 0$, the $-$ sign becomes a $+$. To equate this solution to the others, let $x > 0$ and $\sinh a = 1/\sinh x$, so our new solution is $-a$. One deduces easily that $\cosh a = \cosh x/\sinh x$, so that

$$e^a = \sinh a + \cosh a = \frac{1 + \cosh x}{\sinh x},$$

and hence

$$e^{-2a} = \frac{\sinh^2 x}{(\cosh x + 1)^2} = \frac{\cosh^2 x - 1}{(\cosh x + 1)^2} = \frac{\cosh x - 1}{\cosh x + 1}.$$

Similarly, for our original integral $\int 1/\cosh x dx$, the substitution $u = 1/\cosh x$ leads to the (admittedly rather ungainly) third solution

$$-\sin^{-1} \frac{1}{\cosh x} + c$$

for $x > 0$, and we again leave it to the reader to show that this expression, with c taken to be $\pi/2$, equates to $\tan^{-1} \sinh x$.

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