

Nick's proof that $\zeta(2) = \pi^2/6$

$$\int_0^\infty \frac{1}{\cosh x} dx = [\tan^{-1} \sinh x]_0^\infty = \frac{\pi}{2}$$

$$\begin{aligned} \frac{\pi^2}{4} &= \left(\int_0^\infty \frac{1}{\cosh x} dx \right)^2 = \int_0^\infty \int_0^\infty \frac{1}{\cosh x \cosh y} dy dx \\ &= \int_0^\infty \int_0^\infty \frac{2}{\cosh(x+y) + \cosh(x-y)} dy dx \end{aligned}$$

Substitute $s = x + y$, $t = x - y$ to get

$$\frac{\pi^2}{4} = \int_0^\infty \int_{-s}^s \frac{1}{\cosh s + \cosh t} dt ds = \int_0^\infty \left[\frac{1}{\sinh s} \log \frac{e^t + e^{-s}}{e^t + e^s} \right]_{t=-s}^{t=s} ds = \int_0^\infty \frac{2 \log \cosh s}{\sinh s} ds$$

Substitute $u = \log \cosh s$ to get

$$\frac{\pi^2}{4} = \int_0^\infty \frac{u}{\sinh u} du$$

By the geometric series $1/\sinh u = 2 \sum_{n=0}^\infty e^{-(2n+1)u}$ so

$$\frac{\pi^2}{8} = \sum_{n=0}^\infty \int_0^\infty u e^{-(2n+1)u} du = \sum_{n=0}^\infty \left[-\frac{u e^{-(2n+1)u}}{2n+1} - \frac{e^{-(2n+1)u}}{(2n+1)^2} \right]_{u=0}^{u=\infty} = \sum_{n=0}^\infty \frac{1}{(2n+1)^2}$$

Denoting $\sum_{n=1}^\infty \frac{1}{n^2}$ by $\zeta(2)$

$$\zeta(2) = \sum_{n=0}^\infty \frac{1}{(2n+1)^2} + \sum_{n=1}^\infty \frac{1}{(2n)^2} = \frac{\pi^2}{8} + \frac{\zeta(2)}{4}$$

So

$$\zeta(2) = \pi^2/6$$

NICK JAMESON

nojameson@hotmail.co.uk

<http://nojameson.net>